

Winter 2007 Brain Ticklers

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1 Problem 1

A population of amoeba starts with one amoeba. If there is a 75 percent probability that an amoeba will survive and split to create two amoeba and a 25 percent probability that it will not survive and if all the amoeba in future generations have the same probability of survival, what is the probability of survival?

Solution to problem 1

Let the probability of extinction of the amoeba be p . After the first generation, this is equal to

$$p = \frac{1}{4} + \frac{3}{4}P(\text{extinction given two amoebas})$$

We note that the new probability of extinction given two amoebas must simply be p^2 , so solving the quadratic gives $p = \frac{1}{3}$. Therefore the probability of survival of the population is $\frac{2}{3}$.

2 Problem 2

What is the largest prime p less than 1000 for which $N = p^3 + 2p^2 + p$ has exactly 42 positive integral factors?

Solution to problem 2

A brute force search is simplest, giving $p = 823$. The Mathematica code used is given

```
f[p_] := p + 2 p + p;  
For[i = 168, i > 3,  
  Module[{p, l},  
    p = Prime[i];  
    l = Length[Divisors[f[Prime[i]]]];  
    Print[ToString[p]<>" "<>ToString[l]];  
    If[l == 42, Break[]];  
  ];  
  i--]
```

3 Problem 3

Solve the multiplication problem $LYNDON \times B = JOHNSON$. Each letter stands for a different digit and there are no leading zeros.

Solution to problem 3

Although no computer assistance is necessary, a straightforward solution is found by considering all permutations of possible digits. From this we find $LYNDON = 570140$, $B = 6$, and $JOHNSON = 3420840$. The Mathematica code to find this is given

```
perms = Permutations[{1, 2, 3, 4, 5, 6, 7, 8, 9, 0}];
Map[
Module[{1L, 1Y, 1N, 1D, 1O, 1B, 1J, 1H, 1S, LYNDON, B, JOHNSON, J},
  1L = #[[1]];
  1Y = #[[2]];
  1N = #[[3]];
  1D = #[[4]];
  1O = #[[5]];
  1B = #[[6]];
  1J = #[[7]];
  1H = #[[8]];
  1S = #[[9]];
  LYNDON = 1N + 1010 + 1001D + 10001N + 100001Y + 1000001L;
  B = 1B;
  JOHNSON = 1N + 1010 + 1001S + 10001N + 100001H + 1000001O + 10000001J;
  If[LYNDON*B == JOHNSON, Print[#]; Break[]]] &
, perms];
```

4 Problem 4

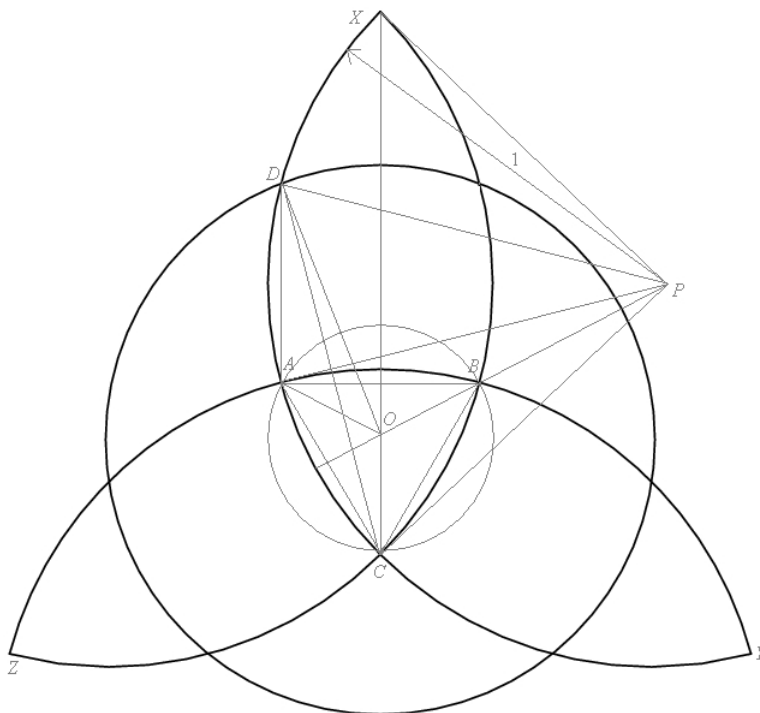
If the radius of the three arcs is 1, find the radius of the central circle such that each arc is divided into 5 equal segments.

Solution to problem 4

We are not given any information regarding the arcs, radius to their centers and their angles will be found first. From the figure, note that $\angle XCA = 30^\circ$ and is an inscribed angle, so the central angle $\angle APX = 60^\circ$. Since we require 5 equal segments on an arc, this means the arc subtends a total of 150° . The radius from the center of the figure to the center of the arcs (\overline{OP}) can be found with the law of sines on $\triangle XOP$:

$$\frac{\overline{XP}}{\sin \angle XOP} = \frac{1}{\sin 60^\circ} = \frac{\overline{OP}}{\sin 45^\circ} = \frac{\overline{OP}}{\angle OXP}$$

so then $\overline{OP} = \sqrt{\frac{2}{3}}$.



To find the radius of the circle \overline{OD} , we will use the law of cosines on $\triangle DAO$:

$$\overline{OD}^2 = \overline{AO}^2 + \overline{DA}^2 - 2\overline{AO}\overline{DA}\cos 120^\circ$$

First we need the radius of the inner circle \overline{AO} . By the law of cosines on $\triangle AOP$,

$$\overline{AO}^2 = \overline{OP}^2 + 1 - 2\overline{OP}\cos 15^\circ$$

We also notice that $\overline{DA} = \overline{AC} = \sqrt{3}\overline{AO}$. These three equations can be solved to obtain

$$\overline{OD} = \sqrt{\frac{5}{3} - \frac{2}{\sqrt{3}}} \approx 0.7155$$

5 Problem 5

Find 3 different three-digit numbers which contain at least one digit 3, such that 3 of the statements below are true and the other 3 false.

- (a) The number is prime.
- (b) The number is a cube of an integer.
- (c) The middle digit is the average of the other two.

- (d) The unit's digit differs from the ten's digit by 3.
- (e) The number has a two-digit prime factor, the digits of which differ by 3 or whose sum is a cube.
- (f) The number is a triangular number.

Solution to problem 5

We notice first that condition (a) that the number is prime must contradict (b), (e), and (f) obviously, and contradicts (c) since it must have 3 as a factor. Therefore, (a) must be false always. The only candidate number satisfying (b) is 343, but fails the other conditions, so condition (b) is also always false. There are only 4 sets of numbers satisfying the remaining 4 conditions, of which 3 must be true for the numbers we seek. It is trivial to generate these sets and compare the intersections, so a brute force search finds 136 satisfies (d), (e), and (f). 369 satisfies (c), (d), and (e). Finally, 630 satisfies (c), (d), and (f).

The Mathematica code used is given.

```
listC = {};
For[i = 103, i <= 993,
  digits = IntegerDigits[i];
  If[MemberQ[digits, 3],
    If[digits[[1]] + digits[[3]] == 2 digits[[2]],
      AppendTo[listC, i]];
  ];
  i++];
listD = {};
For[i = 103, i <= 993,
  digits = IntegerDigits[i];
  If[MemberQ[digits, 3],
    If[Abs[digits[[3]] - digits[[2]]] == 3,
      AppendTo[listD, i]];
  ];
  i++];
listE = {};
For[i = 103, i <= 993,
  digits = IntegerDigits[i];
  If[MemberQ[digits, 3],
    Factors = Map[First, FactorInteger[i]];
    Map[
      If[# > 9 && # < 100,
        If[Abs[IntegerDigits[#][[1]] - IntegerDigits[#][[2]]] == 3,
          AppendTo[listE, i],
          If[IntegerDigits[#][[1]] + IntegerDigits[#][[2]] == 8 ||
            IntegerDigits[#][[1]] + IntegerDigits[#][[2]] == 1,
            AppendTo[listE, i]
          ]
        ]
    ]
  ];
  i++];
```

```

    ]
    ] &
    , Factors];
];
i++];
listF = {};
For[i = 14, i <= 44,
  n = i(i + 1)/2;
  digits = IntegerDigits[n];
  If[MemberQ[digits, 3],
    AppendTo[listF, n]
  ];
  i++];
listC \[Intersection] listD \[Intersection] listE
listC \[Intersection] listD \[Intersection] listF
listD \[Intersection] listE \[Intersection] listF

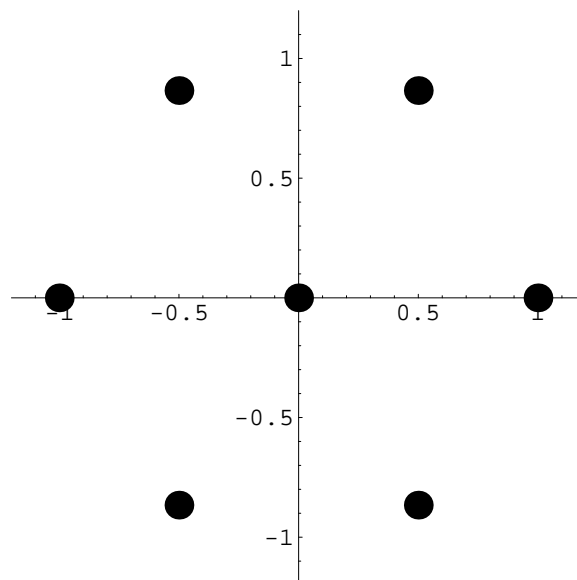
```

6 Bonus

Place 7 distinct points on a plane so that of any 3, at least 2 are exactly 1 cm apart.

Solution to Bonus

The ordinary hexagonal packing of points satisfies the criteria, in which points are placed at the vertices of a perfect hexagon with side length 1 cm, and the seventh point is at the center of the hexagon.



7 Computer Bonus

Find the digits to the problem $TAU^{BETA} = PI$ so that PI is closest to an integer.

Solution to Computer Bonus

Once again, a brute force search finds the answer. If we allow conventional rounding,

$$659^{4365} \approx 17$$

If we only allow rounding down,

$$347^{5034} \approx 19$$

The Mathematica code for the first case is given.

```
perms = Permutations[{1, 2, 3, 4, 5, 6, 7, 8, 9, 0}];
MinDiff = 1;
Saved = {};
Map[
  Module[{lT, lA, lU, lB, lE, lP, lI, TAU, BETA, PI, Num, Diff},
    lT = #1[[1]];
    lA = #1[[2]];
    lU = #1[[3]];
    lB = #1[[4]];
    lE = #1[[5]];
    lP = #1[[6]];
    lI = #1[[7]];
    If[lT != 0 && lP != 0 && lA != 0,
      TAU = 100 lT + 10 lA + lU;
      BETA = 0.1 lB + 0.01 lE + 0.001 lT + 0.0001 lA;
      PI = 10 lP + lI;
      Num = TAU^BETA - PI;
      Diff = Abs[Num];
      If[Diff < MinDiff,
        Print[#1];
        MinDiff = Diff;
        Print[MinDiff];
        Saved = #1;
      ]
    ]
  ] &
, perms;
]]
```