SYMMETRIC AUTONOMOUS DIFFERENTIAL EQUATIONS AND HYPERBOLIC FORMS OF THE LOGISTIC EQUATION

VICTOR LIU

Abstract. Exponential, simple inhibited, logistic, and second-order growth models are often used for describing real world processes. The equations of these four models are varied and do not demonstrate the relationship between them. This paper presents symmetric autonomous differential equations and similarly symmetric implicit solutions to these models, demonstrating their relationship to one another and revealing a deeper significance to the logistic and second-order growth models. Additionally, this paper explores the relationship between the logistic equation and the hyperbolic tangent and cotangent functions, and how to convert to and from both forms of the logistic equation easily.

1. Introduction

This paper presents the study of four types of differential equations that are frequently used for modeling real world growth processes. These are:

1. Exponential growth and decay
2. Simple inhibited growth
3. Logistic growth
4. Second order chemical reaction

The last model name, second order chemical reaction, is not as common as the other three names. The paper adopts it because the model is mostly used in the description of such chemical reaction processes.

The paper shows that the four model equations can be all written in implicit, highly symmetric forms. It is the first time such efforts are made to derive a unified expression for these model equations despite their apparent differences in conventional formulas. The implicit solution forms can also facilitate the development of certain model parameter expressions. Derivations of the implicit forms are based on the simple method of variable substitution. The paper also uses function curve translation to prove that the logistic solution is just another form of the hyperbolic tangent and cotangent functions.

The contents of the paper are organized as the following. The first section describes the four models and their main applications, introduces the model equations and their explicit solutions. The second section derives the implicit model equations and explains the significance and benefits of such equation forms. The third section describes the relationship between the logistic solution and the hyperbolic tangent and cotangent functions. The derived model equations are summarized in the tables at the end of the paper.

2. Growth Modeling Equations

2.1. **Exponential Growth and Decay.** Exponential growth and decay can be represented by one of the basic forms of separable differential equations:

$$\frac{dP}{dt} = rP \tag{2.1}$$

Equation 2.1 states that quantity P varies at a rate directly proportional to the value of P. In the equation r is the rate constant. When $r > 0$, the equation represents exponential growth; when $r < 0$, it represents exponential decay. Equation 2.1 can be easily solved by separation of variables. Assuming that the initial condition P_0 occurs at $t = 0$, then the solution is:

$$P = P_0 e^{rt} \tag{2.2}$$

Exponential growth and decay are most commonly used to model the change of a population, as the rate of population growth or decay is usually proportional to the population itself. For example, as a bacteria population increases, there are more individuals capable of reproduction so the rate of increase also increases. The decay of a radioactive substance also follows this law. As the amount of substance decreases, the rate of decay also decreases. Exponential growth and decay can also be used to model an investment with interest compounded continuously, the processes of a solution being diluted by fresh water, a capacitor being discharged, and many other cases.

2.2. **Simple Inhibited Growth.** Simple inhibited growth assumes that a natural maximum exists such that the growth of a quantity cannot occur beyond it. The equation of simple inhibited growth states that quantity P varies at a rate directly proportional to the difference between this maximum amount, K, and P itself:

$$\frac{dP}{dt} = r\left(K - P\right) \tag{2.3}$$

From equation 2.3 it can be seen that the rate of growth of P is limited by constant K, which is assumed to be positive in this paper. If P starts less than K, the growth of P is positive (assuming $r > 0$) until P is equal to K, at which point the growth of P diminishes to 0. If P is greater than K, then the growth of P is negative, which means P will decrease until it reaches K. Therefore K is often called the **carrying capacity** or **equilibrium value** of the model. Equation 2.3 can also be solved by separation of variables. With the same initial condition as above, the solution is:

$$P = K - \left(K - P_0\right) e^{-rt} \tag{2.4}$$

Simple inhibited growth can model the sales of a newly advertised product, in which case there exists a maximum limit of the product sales. It can also model the processes of an object cooling down to a certain temperature or being dropped from a certain height with air resistance. Other cases include the processes of a solution being diluted by another of different concentration, a capacitor being charged, and certain learning patterns.

2.3. **Logistic Growth.** The previous two models either assume that quantity P varies at a rate directly proportional to itself (P), or to its remaining room for growth ($K - P$). Logistic growth assumes that the growth rate of P is directly proportional to both of these quantities:

$$\frac{dP}{dt} = r' P \left(K - P\right) \tag{2.5}$$

Here r' is used because the logistic equation is more commonly written in this form:

$$\frac{dP}{dt} = rP \left(1 - \frac{P}{K}\right) \tag{2.6}$$

where $r = r'K$. In the above equation, K is the same carrying capacity for P as described before. The constant r is called **intrinsic growth rate**, or the growth rate in the absence of any limiting factors. Equation 2.6 shows that if P is small relative to the carrying capacity K, the rate of its growth will be close to the constant rate r of the exponential growth model. As P nears K, the rate will shrink toward 0, resulting in an S-shaped curve. According to this model, when P reaches K, the growth rate is 0, and the population will be stable. If P were to somehow exceed K, the rate would become negative and the population would decrease toward K.

Equation 2.6 is usually solved by separation of variables with partial fraction, and the solution, with the same initial condition as before, is:

$$P = \frac{KP_0}{P_0 + \left(K - P_0\right) e^{-rt}} \tag{2.7}$$

The logistic equation is mostly used to provide a more realistic model for population growth. It is also frequently used for describing the spreading of diseases or rumors, the autocatalytic chemical reactions, and other processes.

2.4. **Second Order Chemical Reaction.** The second order chemical reaction studied in this paper is represented by the following equation:

$$\frac{dP}{dt} = r \left(a - P\right) \left(b - P\right) \qquad \left(a \neq b\right) \tag{2.8}$$

The model assumes that the rate of change of quantity P is directly proportional to its remaining room for growth within the limits of a and b. From equation 2.8 it can be seen that if $r > 0$, P will increase as long as it is less than both a and b. As P approaches the smaller of a and b, its rate of change will approach 0 and its growth diminishes. In the case of second order chemical reaction, quantity P is the amount of a product generated from two reactants A and B having different initial quantities a and b, respectively. The reaction rate, $\frac{dP}{dt}$, is directly proportional to the remaining quantities of A and B.

Although slightly complicated, Equation 2.8 can also be solved by separation of variables with partial fraction and the solution is:

$$P = a + \frac{b-a}{1 + \left(\frac{b-P_0}{P_0-a}\right) e^{r(b-a)t}} \tag{2.9}$$

3. Implicit Equation Forms

3.1. **Derivation of Implicit Forms.** The four types of growth modeling equations listed in the previous section can be written in implicit, highly symmetric forms [1] for easy memorization and comparison. Derivations of the implicit equation forms are all based on the technique of variable substitution which is commonly used in solving differential equations.

The implicit model equations are similar in format to the exponential growth and decay equation and its solution:

$$\frac{dP}{dt} = rP \tag{3.1}$$

$$P = P_0 e^{rt} \tag{3.2}$$

To convert the simple inhibited growth equation and its solution into implicit forms, make the substitution $Z = K - P$, then $\frac{dZ}{dt} = -\frac{dP}{dt}$. Equation 2.3 becomes $\frac{dZ}{dt} = -rZ$. This new equation of Z resembles the form of equation 2.1 except for the rate constant being $-r$. Therefore Z can be directly solved by using equation 2.2:

$$Z = Z_0 e^{-rt}$$

where $Z = K - P$ and $Z_0 = K - P_0$. Substituting these into the above equations of Z to get:

$$\frac{d\,(K-P)}{dt} = -r\,(K-P) \tag{3.3}$$

$$K - P = (K - P_0)\, e^{-rt} \tag{3.4}$$

For the logistic growth equation and its solution, make the substitution $Z = \frac{K-P}{P}$, then $P = \frac{K}{Z+1}$, and also $\frac{dZ}{dt} = -\frac{K}{P^2}\frac{dP}{dt}$ or $\frac{dP}{dt} = -\frac{P^2}{K}\frac{dZ}{dt}$. Substitute these into equation 2.6 and rearrange:

$$-\frac{P^2 dZ}{K dt} = rP\left(1 - \frac{P}{K}\right)$$

$$\frac{dZ}{dt} = -\frac{rK}{P}\left(1 - \frac{P}{K}\right)$$

$$\frac{dZ}{dt} = -r\,(Z+1)\left(1 - \frac{1}{Z+1}\right)$$

$$\frac{dZ}{dt} = -rZ$$

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[1] Dr. Kuo Chen, Principal of Olympia Institute in San Francisco, summarized the first three growth equations into implicit forms. A portion of this section is adapted from his lecture notes.

Again, the solution of Z can be directly obtained as:

$$Z = Z_0 e^{-rt}$$

Substituting $Z = \frac{K-P}{P}$ and $Z_0 = \frac{K-P_0}{P_0}$ into the above two equations of Z,

Theorem 3.1. *The implicit form of the logistic growth differential equation is*

$$\frac{d}{dt}\left(\frac{K-P}{P}\right) = -r\left(\frac{K-P}{P}\right)$$

and its implicit solution is

$$\frac{K-P}{P} = \frac{K-P_0}{P_0} e^{-rt}$$

The exponential growth and decay equation and solution can also be written in implicit forms so that they match the other two growth equations in format. Let $Z = \frac{1}{P}$, then $\frac{dZ}{dt} = -\frac{1}{P^2}\frac{dP}{dt}$ or $\frac{dP}{dt} = -P^2\frac{dZ}{dt}$. Substitute these into equation 2.1:

$$-P^2\frac{dZ}{dt} = rP$$

$$\frac{dZ}{dt} = -r\frac{1}{P} = -rZ$$

The solution of Z is:

$$Z = Z_0 e^{-rt}$$

Substituting $Z = \frac{1}{P}$ and $Z_0 = \frac{1}{P_0}$ in the above equations of Z to get:

$$\frac{d}{dt}\left(\frac{1}{P}\right) = -r\left(\frac{1}{P}\right) \tag{3.5}$$

$$\frac{1}{P} = \frac{1}{P_0}e^{-rt} \tag{3.6}$$

The second order chemical reaction equation and its solution can be converted into implicit forms by making the substitution $Z = \frac{b-P}{a-P}$, then $aZ - PZ = b - P$ or $P = \frac{aZ-b}{Z-1}$. The differentials become:

$$\frac{dZ}{dt} = \frac{-(a-P)+(b-P)}{(a-P)^2}\frac{dP}{dt} = \frac{b-a}{(a-P)^2}\frac{dP}{dt}$$

$$\frac{dP}{dt} = \frac{(a-P)^2}{b-a}\frac{dZ}{dt}$$

Substituting all of these into equation 2.8 to obtain

$$\begin{aligned}\frac{(a-P)^2}{b-a}\frac{dZ}{dt} &= r\left(a - \frac{aZ-b}{Z-1}\right)\left(b - \frac{aZ-b}{Z-1}\right)\\ &= r\left(\frac{aZ-a-aZ+b}{Z-1}\right)\left(\frac{bZ-b-aZ+b}{Z-1}\right)\\ &= r\left(\frac{b-a}{Z-1}\right)\left(\frac{bZ-aZ}{Z-1}\right)\end{aligned}$$

$$= r\frac{Z\left(b-a\right)^2}{\left(Z-1\right)^2}$$

Since $a - P = a - \frac{aZ-b}{Z-1} = \frac{aZ-a-aZ+b}{Z-1} = \frac{b-a}{Z-1}$, the expression $\left(a-P\right)^2$ on the left side of the above equation can be replaced:

$$\frac{\left(\frac{b-a}{Z-1}\right)^2}{b-a}\frac{dZ}{dt} = r\frac{Z\left(b-a\right)^2}{\left(Z-1\right)^2}$$

$$\frac{1}{b-a}\frac{dZ}{dt} = rZ$$

$$\frac{dZ}{dt} = r\left(b-a\right)Z$$

Since the expression $r\left(b-a\right)$ is a constant, the solution for Z is:

$$Z = Z_0 e^{r(b-a)t}$$

Substituting $Z = \frac{b-P}{a-P}$ and $Z_0 = \frac{b-P_0}{a-P_0}$ into the above equations of Z to get:

$$\frac{d}{dt}\left(\frac{b-P}{a-P}\right) = r\left(b-a\right)\left(\frac{b-P}{a-P}\right) \qquad (a \neq b) \tag{3.7}$$

$$\frac{b-P}{a-P} = \left(\frac{b-P_0}{a-P_0}\right)e^{r(b-a)t} \tag{3.8}$$

3.2. **Significance of Implicit Forms.** The development of the implicit forms is the first such attempt to systematically derive a common expression for these four types of diversified model equations. The method used in the derivations is conventional - the commonly used substitution of variables in solving differential equations, no special skills required. The implicit forms clearly demonstrate the nature of these models. For example, when $r > 0$, the simple inhibited growth is an exponential decay of the $(K-P)$ quantity, the logistic growth is an exponential decay of the $\left(\frac{K-P}{P}\right)$ quantity, and the second order chemical reaction is an exponential growth of the $\frac{b-P}{a-P}$ quantity if $b > a$, or exponential decay if $b < a$.

The implicit equation forms are simple and highly symmetric. They can be easily compared and memorized. The implicit forms can often provide a quicker way to solve the more complicated logistic equation problems, as demonstrated by Example 3.1 below. The implicit model equations can also facilitate the development of expressions of certain model parameters that are difficult to obtain from the explicit formulas. This is evidenced by following Examples 3.2 and 3.3.

Example 3.2. *A contagious disease is spreading in a town of 6000 people. Initially, three people are infected; 3 days later it spreads to 300 people. Suppose the disease spreads at a rate proportional to both the infected people and the unaffected people. Find the number of days for the disease to spread to 50% of the population.*

a) Use explicit solution 2.7 to solve for e^{-r} with the known information $P_0 = 3$, $K = 6000$ and $P(3) = 300$.

$$300 = \frac{(6000)(3)}{3 + (6000-3)\,e^{-3r}}$$

$$3+(6000-3)\,e^{-3r}=60$$
$$1999e^{-3r}=19$$
$$e^{-r}=\left(\frac{19}{1999}\right)^{\frac{1}{3}}$$

Now use the value of e^{-r} and $P=3000$ to solve for t:

$$3000=\frac{18000}{3+5997e^{-rt}}$$
$$3+5997e^{-rt}=6$$
$$1999e^{-rt}=1$$
$$\left(\frac{19}{1999}\right)^{\frac{t}{3}}=\frac{1}{1999}$$
$$t=\frac{3\ln\frac{1}{1999}}{\ln\frac{19}{1999}}\approx 4.9\text{ days}$$

b) Use implicit solution **??** with conditions $P_0=3$, $K=6000$ and $P\,(3)=300$:

$$\frac{6000-300}{300}=\frac{6000-3}{3}e^{-r(3)}$$
$$e^{-r}=\left(\frac{19}{1999}\right)^{\frac{1}{3}}$$

Solving for t when $P\,(t)=3000$:

$$\frac{6000-3000}{3000}=\frac{6000-3}{3}e^{-rt}$$
$$1=1999\left(\frac{19}{1999}\right)^{\frac{t}{3}}$$
$$t=\frac{3\ln\frac{1}{1999}}{\ln\frac{19}{1999}}\approx 4.9\text{ days}$$

Example 3.3. *The following table shows the fish population in a lake in three consecutive decades. Assume that the population grows logistically. Estimate the lake's maximum capacity for the fish population.*

Year	*1960*	*1970*	*1980*
Population (thousands of fish)	*61.3*	*72.8*	*84.0*

In this problem, the known conditions are $P_0=61.3$, $P_1=72.8$, $P_2=84.0$, $t_0=0$, $t_1=1$, $t_2=2$ and the unknown is K. The method is to derive a general expression for K first, then plug in the known numbers to calculate its value. According to implicit logistic solution **??**:

$$\frac{K-P_1}{P_1}=\left(\frac{K-P_0}{P_0}\right)e^{-rt_1}\text{ and }\frac{K-P_2}{P_2}=\left(\frac{K-P_0}{P_0}\right)e^{-rt_2}$$

Divide the first equation by the second:

$$\frac{(K-P_1)\,P_2}{(K-P_2)\,P_1}=e^{r(t_2-t_1)}$$

Rearranging the first equation:

$$\frac{(K-P_0)\,P_1}{(K-P_1)\,P_0}=e^{rt_1}$$

Since $t_2 - t_1 = t_1 = 1$, or $e^{rt_1} = e^{r(t_2-t_1)}$, the above two equations can be equated:

$$\frac{(K-P_0)\,P_1}{(K-P_1)\,P_0} = \frac{(K-P_1)\,P_2}{(K-P_2)\,P_1}$$

Solving the above quadratic equation of K:

$$(K-P_0)\,(K-P_2)\,P_1^2 = (K-P_1)^2\,P_0P_2$$

$$\left(P_1^2 - P_0P_2\right)K^2 + \left(2P_0P_1P_2 - P_0P_1^2 - P_2P_1^2\right)K = 0$$

$$K_1 = 0,\ K_2 = \frac{P_1\,(P_1P_0 + P_1P_2 - 2P_0P_2)}{P_1^2 - P_0P_2}$$

Obviously, $K_1 = 0$ is not a meaningful solution, so the solution is:

$$K = \frac{72.8\,(72.8\,(61.3) + 72.8\,(84.0) - 2\,(61.3)\,(84.0))}{72.8^2 - (61.3)\,(84.0)} \approx 135.0 \text{ thousand}$$

Example 3.4. *The census taken in 1990 and 1994 of a city's population showed that it had 2.48 million and 2.67 million residents, respectively. Assume that the maximum capacity of the city is 3.20 million and the population increases logistically. Estimate the city's population in 1978.*

In this problem the known conditions are $P_1 = 2.48$, $P_2 = 2.67$, $t_1 = 12$, $t_2 = 16$, $K = 3.20$, and the unknown is P_0. The same method is used to derive a general solution of P_0 first, then to plug in the known number to solve for its value. From the previous example:

$$\frac{(K-P_0)\,P_1}{(K-P_1)\,P_0} = e^{rt_1} \text{ and } \frac{(K-P_1)\,P_2}{(K-P_2)\,P_1} = e^{r(t_2-t_1)}$$

Solve for e^r in both equations and equate them:

$$\left(\frac{(K-P_0)\,P_1}{(K-P_1)\,P_0}\right)^{\frac{1}{t_1}} = \left(\frac{(K-P_1)\,P_2}{(K-P_2)\,P_1}\right)^{\frac{1}{t_2-t_1}}$$

$$\frac{(K-P_0)\,P_1}{(K-P_1)\,P_0} = \left(\frac{(K-P_1)\,P_2}{(K-P_2)\,P_1}\right)^{\frac{t_1}{t_2-t_1}}$$

$$\frac{K-P_0}{P_0} = \frac{K-P_1}{P_1}\left(\frac{(K-P_1)\,P_2}{(K-P_2)\,P_1}\right)^{\frac{t_1}{t_2-t_1}}$$

$$\frac{K}{P_0} = 1 + \frac{K-P_1}{P_1}\left(\frac{(K-P_1)\,P_2}{(K-P_2)\,P_1}\right)^{\frac{t_1}{t_2-t_1}}$$

$$P_0 = \frac{K}{1 + \frac{K-P_1}{P_1}\left(\frac{(K-P_1)P_2}{(K-P_2)P_1}\right)^{\frac{t_1}{t_2-t_1}}}$$

So the final solution is:

$$P_0 = \frac{3.20}{1 + \frac{3.20-2.48}{2.48}\left(\frac{(3.20-2.48)2.67}{(3.20-2.67)2.48}\right)^{\frac{12}{4}}} \approx 1.68 \text{ million}$$

References

1. D. M. Bradley. Verhulst's Logistic Curve. *The College Mathematics Journal*, 32:94-98, 2001.