The four mice pursuit curve solution using Laplace Transforms

Victor Liu

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Abstract

A solution to the classic four mice pursuit problem is presented using the Laplace Transform method.

The problem

Suppose four mice are located at each of the four corners of a square table, and each mouse runs toward the one to its right. Find the parametric curves that describe the trajectory of the mice. Realize of course, that the velocity of each mouse does not matter, they will still follow the same trajectory.

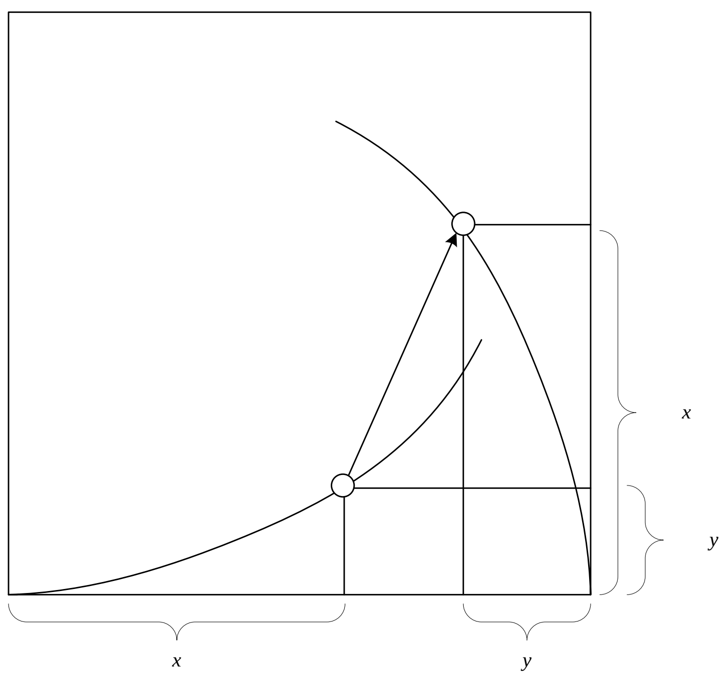

Figure 1: A plot of the solution, actually.

The solution

We begin with a diagram of the mice on the table. Let the table be of size unity on each side and we concentrate on the bottom left mouse. We can express the slope of its present velocity vector as

$$\frac{dy}{dx} = \frac{x - y}{1 - x - y} \tag{1}$$

To introduce the parametric velocities, we can instead write

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{x-y}{1-x-y} \tag{2}$$

Or splitting it up

$$\frac{dy}{dt} = x - y \tag{3}$$

$$\frac{dx}{dt} = 1 - x - y \tag{4}$$

Laplace transforming this set, and using the initial conditions $y(0) = 0$ and $x(0) = 0$,

$$sY(s) = X(s) - Y(s) \tag{5}$$

$$sX(s) = \frac{1}{s} - X(s) - Y(s) \tag{6}$$

Solving for $X(s)$ using the first equation, we have $X(s) = (s+1)Y(s)$. Substituting this into the second,

$$s(s+1)Y(s) = \frac{1}{s} - (s+1)Y(s) - Y(s) \tag{7}$$

Solving for $Y(s)$, and by a similar process for $X(s)$, we arrive at

$$Y(s) = \frac{1}{s(s^2+2s+2)} = \frac{1}{2s} + \frac{-\frac{1}{2}s - 1}{s^2+2s+2} = \frac{1}{2s} - \frac{1}{2}\frac{s-(-1)}{[s-(-1)]^2+1} - \frac{1}{2}\frac{1}{[s-(-1)]^2+1} \tag{8}$$

$$X(s) = \frac{s+1}{s(s^2+2s+2)} = \frac{1}{2s} + \frac{-\frac{1}{2}s}{s^2+2s+2} = \frac{1}{2s} - \frac{1}{2}\frac{s-(-1)}{[s-(-1)]^2+1} + \frac{1}{2}\frac{1}{[s-(-1)]^2+1} \tag{9}$$

The partial fractions and subsequent manipulations was to form the equations into forms resembling the transforms. Refer to the appendix for the relevant transforms. Inverse transforming this set,

$$y(t) = \frac{1}{2} - \frac{1}{2}e^{-t}\cos t - \frac{1}{2}e^{-t}\sin t \tag{10}$$

$$x(t) = \frac{1}{2} - \frac{1}{2}e^{-t}\cos t + \frac{1}{2}e^{-t}\sin t \tag{11}$$

And we are done. In this process, no calculus has been employed. The hardest part was the algebraic manipulation or looking up the transforms in a table. We may find the others by the same process, but using a different initial condition. If $y(0) = 1$, this would make $y'(t) \to sY(s) - 1$, etc. If a geometric or symmetry argument is used, then each successive trajectory has as its y-component the current x-component, as as its x-component one minus the current y-component. To write it all out,

$$x_1(t) = \frac{1}{2} - \frac{1}{2}e^{-t}\cos t + \frac{1}{2}e^{-t}\sin t \tag{12}$$

$$y_1(t) = \frac{1}{2} - \frac{1}{2}e^{-t}\cos t - \frac{1}{2}e^{-t}\sin t \tag{13}$$

$$x_2(t) = \frac{1}{2} + \frac{1}{2}e^{-t}\cos t + \frac{1}{2}e^{-t}\sin t \tag{14}$$

$$y_2(t) = \frac{1}{2} - \frac{1}{2}e^{-t}\cos t + \frac{1}{2}e^{-t}\sin t \tag{15}$$

$$x_3(t) = \frac{1}{2} + \frac{1}{2}e^{-t}\cos t - \frac{1}{2}e^{-t}\sin t \tag{16}$$

$$y_3(t) = \frac{1}{2} + \frac{1}{2}e^{-t}\cos t + \frac{1}{2}e^{-t}\sin t \tag{17}$$

$$x_4(t) = \frac{1}{2} - \frac{1}{2}e^{-t}\cos t - \frac{1}{2}e^{-t}\sin t \tag{18}$$

$$y_4(t) = \frac{1}{2} + \frac{1}{2}e^{-t}\cos t - \frac{1}{2}e^{-t}\sin t \tag{19}$$

The plot of the solution is given below.

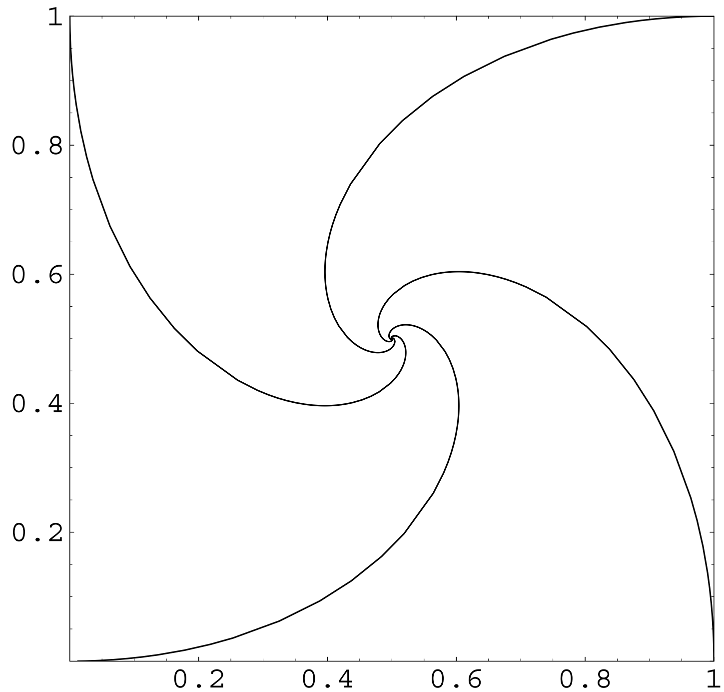

Figure 2: A plot of the solution curves.

Appendix of Laplace Transforms

From t space to s space,

$$\mathcal{L}\left\{f'(t)\right\} = sF(s) - f(0)$$

$$\mathcal{L}\left\{e^{at}\cos kt\right\} = \frac{k}{(s-a)^2+k^2}$$

$$\mathcal{L}\left\{e^{at}\sin kt\right\} = \frac{s-a}{(s-a)^2+k^2}$$